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C^1 RATIONAL FINITE ELEMENTS

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Abstract—There is a well-defined construction of rational basis functions for patchwork C^0 approximation over a regular algebraic partition of a planar region [2]. In this note the construction is generalized to C^1 approximation for convex quadrilateral elements.

1. C^1 APPROXIMATION OVER TRIANGLES

It is well known that the C^1 approximation over a triangulation may be achieved with degree-5 polynomials and 21 degrees of freedom over a triangle. The Clough-Tocher macrotriangle element yields C^1 with piecewise cubic approximation over each triangle and only 18 degrees of freedom. A degree-3 C^0 triangle basis has 10 degrees of freedom (where one obtains the complete cubic by fitting value and gradient at the vertices and value at an interior point usually chosen at the triangle centroid). The normal derivative along each triangle edge is quadratic with value fit at the two vertices. The normal derivatives on adjacent triangles require one more constraint to assure the C^1 approximation. Ignoring this discontinuity of the derivatives in applications demanding C^1 approximation is a "variational crime" that has paid in some cases. More law-abiding practitioners have appended three rational basis functions to the usual cubic basis to yield the required C^1 approximation. Numerical studies have shown that the extra computational effort often does not justify this refinement and that here crime does pay [3]. Strang and Fix are "opposed" to the rational supplement because of the integration complexity, but they err in asserting that this does not achieve degree-3 polynomial approximation over the triangle [4, p. 82].

The notation in [2] will be used. The symbol $(m;n)$ denotes both the line through points m and n and the linear form that vanishes on this line. Higher-order curves may be denoted by symbols $(a;\ b;\ c;\ \dots)$ where the indicated nodes lie on the curve, but where auxiliary data may be required to specify the curve uniquely. Let the triangle vertices be 1, 2 and 3 and let $(1;\ 2;\ 3)$ denote its circumcircle. Then the rational basis function which ties down the quadratic variation of the normal derivative on side $(1;2)$ is

$$S = \frac{(1;2)\,(2;3)^2\,(3;1)^2}{(1;2;3)}. \tag{1}$$

It is easily shown that the normal derivative of S is zero on sides $(2;3)$ and $(3;1)$ and quadratic on side $(1;2)$. A cricual algebraic-geometry theorem for this analysis is [2, Theorem 4.13]:

THEOREM 1. *If curves P and R do not contain the irreducible curve Q and the divisor of P on Q is equal to the divisor of R on Q, then P is congruent to R on Q.*

The "divisor" is a generalization of the "set of intersections," and P congruent to R on Q means that there is a constant c such that $P - cR = 0$ on all of curve Q. The theorem is

This analysis was presented at a SIAM conference on applied geometry held at RPI in July of 1987. That presentation included reference to C^1 approximation of degree-2 over a triangle which was shown to be incorrect by P.L. Powar of R.D. University in Jabalpur. She and R. Rao describe the construction and a counterexample in a companion paper in this journal [1].

illustrated by considering S on line $(1;2)$. The symbol $A.B$ denotes the intersection set of A and B. One has $(2;3)(3;1).(1;2) = 1,2$ and $(1;2;3).(1,2) = 1,2$. It follows from Theorem 1 that the normal derivative of S varies like $S' = c\,(2;3)(3;1)$ on side $(1;2)$. Theorem 1 provides a means for cancellation of factors in numerator and denominator of rational functions restricted to curves.

Another theorem essential for this analysis is:

THEOREM 2. *If polynomial $P(x,y)$ vanishes everywhere on the irreducible curve G (on which $G(x,y) = 0$), then G is a factor of P. If the derivative of P normal to curve G is also zero everywhere on G, then G^2 is a factor of P.*

It will now be shown that the C^1 triangle with the 10 polynomial and 3 rational basis functions does achieve degree-3 polynomial approximation. The difference between a cubic and its interpolant with these 13 basis functions vanishes on the triangle boundary as does its normal derivative. This difference is a polynomial of maximal degree 5 divided by the quadratic $(1;2;3)$. The square of the boundary is a polynomial of degree 6 which must divide the fifth degree numerator (Theorem 2). This is possibly only if the numerator is the zero polynomial. This analysis shows that the basis function associated with the interior node is extraneous. Degree-3 C^1 approximation is thus achieved with 12 basis functions.

It should be noted that the interior node is needed in the absence of the rational functions to achieve degree-3 C^0 approximation. In the absence of normal derivative matching, the boundary of degree 3 divides the difference polynomial of degree 3, when the difference is a multiple of the cubic that vanishes on the boundary. The interior node basis function is precisely this cubic, and forces the difference polynomial to vanish at the interior node as well as on the boundary, thereby guaranteeing degree-3 approximation.

It is not clear that the difficulties associated with numerical integration cannot be resolved by further analysis, so that these rational C^1 triangles may become a valuable tool. The work of Andersen and McLeod [5] should be reviewed in this context. Possible advantages of a C^1 triangle with only 12 degrees of freedom should not be discarded lightly. C^1 approximation is required for commonly used finite element formulations for solution of differential equations of order four.

2. RECTANGLES

It is well-known that C^1 approximation is achieved over a rectangular grid with a bicubic basis fitting values, gradients, and mixed derivatives at the vertices. However, this basis fails when the vertex angles are not all right angles. A single mixed derivative at each vertex cannot yield continuity of the normal derivatives across all four sides. The basis for C^1 approximation over a rectangle has 16 polynomials functions. A rational alternative that does generalize to convex quadrilaterals will now be developed. Consider first the 13-node C^0 degree-3 element. (Fig. 1).

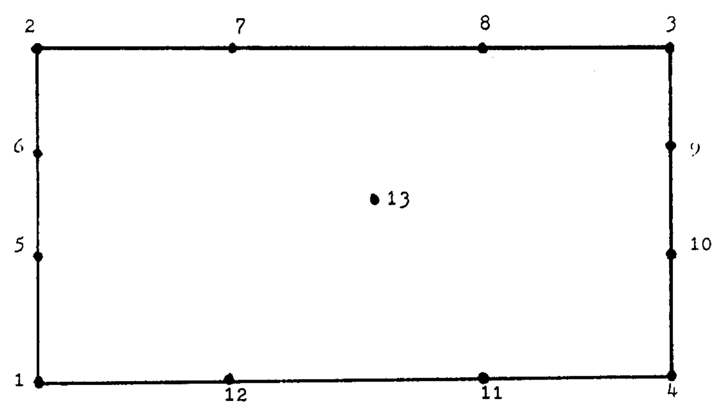

Figure 1. A degree-3 C^0 rectangle.

The basis function associated with node 1 is $W_1 = (2;3)(3;4)(5;6;11;12;13)$ with the last factor, the quadratic which vanishes on the unique conic through the indicated five points. W_1

is normalized to unity at 1. The basis functions associated with nodes 5 and 13 are $W_5 = (2;3)(3;4)(4;1)(6;13)$, and $W_{13} = (1;2)(2;3)(3;4)(4;1)$. The remaining basis functions are of similar form. Each W_i is normalized to unity at i. The normal derivative of any linear combination of these functions is cubic on each side. This basis may be used to fit values and gradients at vertices and the value at the interior node. Thus, to achieve C^1 approximation, 2 additional degrees of freedom must be tied down on each side. For side $(1;2)$, the following function accomplishes this task:

$$V(1,2) = \frac{(2;3)^2\,(3;4)^2\,(4;1)^2\,(1;2)\,(a+b\,y)}{(1;2;3;4)}, \tag{2}$$

where $(1;2;3;4)$ is an ellipse through the vertices.

$V(1,2)$ vanishes along the rectangle perimeter and its normal derivative vanishes on all sides but $(1;2)$, on which $(3;4)$ is constant and $(2;3)(4;1)$ is congruent to $(1;2;3;4)$. Thus, the derivative of $V(1,2)$ normal to side $(1;2)$ is congruent to $(2;3)(4,1)(a+by)$ on side $(1;2)$. Values for a and b may be chosen to fit the normal derivate at the midpoint of side $(1;2)$ and to yield an interpolant with a quadratic normal derivative on side $(1;2)$. The parameters a and b in $V(1,2)$ depend on the vertex data. This is satisfactory for interpolation and surface fitting but not for finite element computation.

Similar functions may be introduced on the other three sides. That degree-3 approximation is achieved is proved by noting that the difference between a cubic and its interpolant is a polynomial of maximal degree 8 divided by a quadratic, and that the numerator must have the square of the fourth degree boundary as a factor. The numerator must also vanish at the interior node. This establishes that the numerator of maximal degree 8 must be the zero polynomial. A total of 17 basis functions are required here. This is a poor alternative to the bicubic polynomial basis and has been introduced only because of its relevance to generation of basis functions for C^1 approximation over arbitrary convex quadrilaterals. It should be noted that even for C^0 degree-3 approximation over a rectangle, an off-boundary node seems needed. Actually, it can be shown by a more subtle argument that degree-3 approximation is achieved without this extra node [6].

Moreover, for a rectangle (as noted previously) one needs only to introduce four V-functions to fit the mixed derivative at each vertex. Thus, instead of the rational V-functions, one may use vertex functions of the form:

$$V(1) = (1;2)\,(4;1)\,(2;3)^2\,(3;4)^2. \tag{3}$$

One may remove the interior node 13 and choose bicubic W-functions. This choice is equivalent to the usual bicubic C^1 approximation, in which combinations of these basis functions orthogonal with respect to the fitted values and derivatives are chosen as a more convenient basis. However, this does not generalize to convex quadrilaterals.

3. CONVEX QUADRILATERALS

Consider a convex quadrilateral with exterior diagonal (A,B). The W-basis for degree-3 C^0 approximation is as for the rectangle, but now divided by the linear factor $(A;B)$ as shown in reference [2]. (Fig. 2)

Computation is simplified by the use of basis functions generated for fitting value and gradient at each vertex. The three functions associated with node 1 are:

$$W(1)_{1,0,0} = \frac{(2;3)\,(3;4)\,(2;4)\,(C;D)}{(A;B)}, \tag{4}$$

with all factors normalized to unity at 1.

Let $d(a,b)$ denote the distance from a to b. Point C is chosen so that $d(1,C) = d(1,4)/2$. Point D is chosen so that $d(1,D) = d(1,2)/2$. This choice of C and D yields a zero gradient for this basis function at node 1.

$$W(1)_{0,1,0} = \frac{(2;3)\,(3;4)\,(2;4)\,(1;E)}{(A,B)}. \tag{5}$$

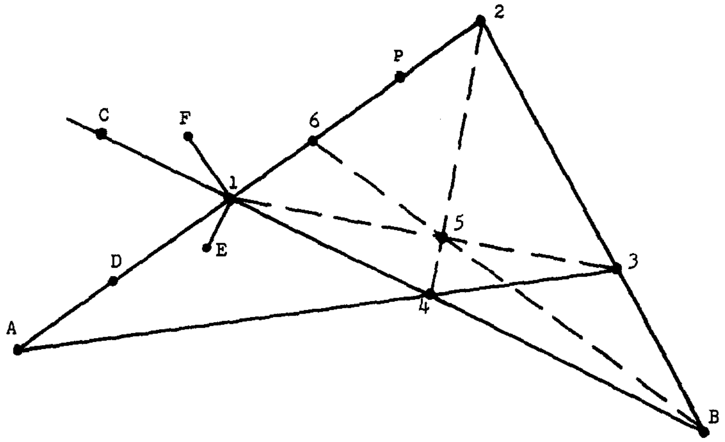

Figure 2. A degree-3 C^0 convex quadrilateral.

Point E is chosen so that line $(1;E)$ is perpendicular to $(1;4)$. The component of the gradient of this basis function at 1, normal to $(1;4)$, is thus zero.

$$W(1)_{0,0,1} = \frac{(2;3)\,(3;4)\,(2;4)\,(1;F)}{(A;B)}. \tag{6}$$

Point F is chosen so that line $(1;F)$ is perpendicular to $(1;2)$. The component of the gradient of this basis function at 1, normal to $(1;2)$, is thus zero.

Analogous functions are generated for vertices 2, 3 and 4. The function associated with node 5, at which the diagonals intersect, is $W_5 = (1;2)(2;3)(3;4)(4;1)/(A;B)$. This function and its gradient vanish at all four vertices. Moreover, its derivative normal to each side is quadratic on the side. The normal derivative of each vertex function varies as a cubic/$(A;B)$ on each side. Thus, the variation on each side of the normal derivative of any linear combination of the 13 basis functions is (at most cubic)/$(A;B)$.

The added V-basis functions are similar to those introduced for the rectangle:

$$V(1,2) = \frac{(1;2)\,(2;3)^2\,(3;4)^2\,(4;1)^2\,(5;P)}{(A;B)^3\,(1;2;3;4)}. \tag{7}$$

The derivative of $V(1,2)$ normal to $(1;2)$ on $(1;2)$ is $(2;3)(4;1)(5;P)/(A;B)$. The choice of P is crucial. The ratio of $(5;P)$ at 6 to $(5;P)$ at A is equal to the ratio of the lengths of the segments $6,P$ and A,P of line $(1;2)$ with appropriate signs. Let this ratio be denoted by $r(P)$. The coefficients of the 17 degree-3 C^0 basis functions are first determined to interpolate values and gradients at the four vertices and the value at node 5. Let the value of the derivative of this C^0 interpolant normal to $(1;2)$ at 6 be $n(6)$, and at A be $n(A)$. Point P is chosen so that $r(P) = -n(6)/n(A)$. (If $N(A) = 0$, then $P = A$.) Then the coefficient of $V(1,2)$ is chosen to interpolate the normal derivative to $(1;2)$ at 6. The interpolation function for the C^1 approximation has a normal derivative on $(1;2)$ which varies as a cubic/$(A;B)$. Node P has been chosen so that the cubic numerator vanishes at A. This allows cancellation of the factor $(A;B)$, yielding quadratic variation of the normal derivative. A similar analysis applies to all four sides of the quadrilateral. It is essential that the behavior not be rational on $(1;2)$, since the exterior diagonal of the adjacent quadrilateral will often not be equal to $(A;B)$.

The difference between the interpolant of a cubic and the cubic is a polynomial of maximal degree 8 divided by $(A;B)^3(1;2;3;4)$. The square of the boundary must be a factor of the numerator, which must also vanish at node 5. Hence, the numerator is the zero polynomial and degree-3 is attained. The variation of the normal derivative is quadratic on each boundary segment with values fitted at the vertices and one side point. Thus, the normal derivative is continuous and C^1 approximation is assured.

4. FINITE ELEMENT APPLICATION: SOLVING A VARIATIONAL CRIME

Integrals of products of basis functions and their derivatives of various orders occur in calculation of finite element equations from which nodal parameters are computed. When computing the convex quadrilateral elements, one cannot use basis functions which depend on the free parameters (nodal values and derivatives) in the variational equations. This problem will be addressed shortly. The V-functions may be used to smooth a C^0 approximation obtained by variational crime. In this application they are used strictly for interpolation and do not appear in the integration stage.

Let u denote the discrete solution to a problem with a nonconforming degree-3 basis over a quadrilateral grid. The components of u are the values and first derivatives at the quadrilateral nodes. The supplementary V-basis functions may be determined with these u components. One may evaluate the normal derivative at a side node with the C^0 expansion in each element sharing the side, and choose the normalization of the corresponding V-basis functions to yield the average of these values in both elements. This yields a C^1 interpolant with the computed u, which is an attempt to "cover up" rather than solve the variational crime.

Of greater interest is true C^1 computation. For this purpose, one may split each V function into two basis functions. For example, define

$$U(1;2) = \frac{(1;2)\,(2;3)^2\,(3;4)^2\,(4;1)^2}{(A;B)^3\,(1;2;3;4)}. \tag{8}$$

Then the V-basis functions along side $(1;2)$ may be chosen as $V_6(1;2) = (5;A)U(1;2)$ and $V_A(1;2) = (5;6)U(1;2)$. The V_6 function is used to fit the normal derivative at 6 and the V_A function is used to equate the numerator of the normal derivative at A to zero, thus reducing the rational variation to quadratic variation. Note that $(5;P)$ is a linear combination of $(5;A)$ and $(5;6)$, so that the combination of V_A and V_6 obtained from the finite element equations yields an interpolant identical to that which would be obtained with the computed parameters and $V(1;2)$. The number of basis functions over a convex quadrilateral for C^1 computation is thus 21 rather than 17. The equation at the interior node can be removed by static condensation.

5. DISCUSSION

The initial work [2] dealt only with a rational basis for C^0 approximation over an algebraic reticulation. Basic theorems in algebraic geometry led to a robust algorithm for generating rational basis functions for any degree of approximation. Application has been sparse as a result of the complexity of such elements and the numerical efficacy of isoparametrics. A major difficulty with rational bases is the need for numerical integration of products of these functions and their derivatives over complicated element domains.

When C^1 approximation is demanded, the elements become even more complex. However, once quadrature formulas are devised for integrating rational functions over element domains, the added complexity of a C^1 over a C^0 approximation is not unreasonable. Alternatives to these rational bases are not known for complex element geometry. Although a means for achieving C^1 approximation has been introduced here, practical implementation has yet to be attempted. This note addresses only theoretical foundations.

Generalization to three space dimensions is also of interest. For example, the bicubic C^1 element for a rectangle does not generalize to cubes. One cannot obtain a tricubic C^1 basis over a cube. However, it may be possible to supplement the tricubic C^0 approximation over a cube with a set of rational basis functions, to fit the normal derivative on each bounding plane so as to achieve C^1 approximation. This is an area for further research.

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